

# Reconstruction of a piecewise smooth absorption coefficient by an acousto-optic process\*

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March 7, 2013

## Abstract

The aim of this paper is to tackle the nonlinear optical reconstruction problem. Given a set of acousto-optic measurements, we develop a mathematical framework for the reconstruction problem in the case where the optical absorption distribution is supposed to be a perturbation of a piecewise constant function. Analyzing the acousto-optic measurements, we establish a new equation in the sense of distributions for the optical absorption coefficient. For doing so, we introduce a weak Helmholtz decomposition and interpret in a weak sense the cross-correlation measurements using the spherical Radon transform. We next show how to find an initial guess for the unknown coefficient and finally construct the true coefficient by providing a Landweber type iteration and proving that the resulting sequence converges to the solution of the system constituted by the optical diffusion equation and the new equation mentioned above. Our results in this paper generalize the acousto-optic process proposed in [4] for piecewise smooth optical absorption distributions.

Mathematics Subject Classification (MSC2000): 35R30, 35B30.

Keywords: acousto-optic inverse problem, spherical Radon transform, Helmholtz decomposition, piecewise smooth functions, reconstruction, Landweber iteration, stability.

## 1 Introduction

Let  $\Omega$  be a bounded  $\mathcal{C}^1$ -domain of  $\mathbb{R}^d$ , where  $d \in \{2, 3\}$ . We denote by  $\nu$  the outward normal to  $\partial\Omega$ , the boundary of  $\Omega$ . We need the following functional spaces. For  $m$  a non-negative integer, we define the space  $H^m(\Omega)$  as the family of all functions in  $L^2(\Omega)$ , whose weak derivatives of orders up to  $m$  also belong to  $L^2(\Omega)$ . For  $m \geq 1$ , the space  $H^{m-1/2}(\partial\Omega)$  denotes the set of the traces on  $\partial\Omega$  of all functions in  $H^m(\Omega)$ . We let  $H_0^m(\Omega)$  be the closure of  $\mathcal{C}_c^\infty(\Omega)$  in  $H^m(\Omega)$ , where  $\mathcal{C}_c^\infty(\Omega)$  is the set of all infinitely differentiable

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\*This work was supported by the ERC Advanced Grant Project MULTIMOD-267184.

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functions with compact supports in  $\Omega$ . We denote by  $H^{-m}(\Omega)$  the dual of  $H_0^m(\Omega)$ . Finally, for  $p \geq 1$ , we introduce  $W^{m,p}(\Omega)$  as the space of functions whose weak derivatives of orders up to  $m$  are functions in  $L^p(\Omega)$  and  $W_0^{m,p}(\Omega)$  to be the closure of  $\mathcal{C}_c^\infty(\Omega)$  in  $W^{m,p}(\Omega)$ . Here,  $L^p(\Omega)$  is defined in the usual way. Note that  $W^{m,2}(\Omega) = H^m(\Omega)$  and  $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ .

Suppose that  $\Omega$  represents an optical medium and let  $a_* : \Omega \rightarrow \mathbb{R}^+$  be the optical absorption coefficient of  $\Omega$ . When the medium  $\Omega$  is illuminated with infrared light spots, the optical energy density  $\Phi_* \in H^2(\Omega)$  inside  $\Omega$  satisfies the diffusion equation

$$\begin{cases} -\Delta \Phi_* + a_* \Phi_* = 0 & \text{in } \Omega, \\ l \partial_\nu \Phi_* + \Phi_* = g & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $l \geq 0$  is the extrapolation length, computed from the radiative transport theory [25], and the illumination function on the boundary  $g \in H^{1/2}(\partial\Omega)$  satisfies  $g \geq 0$  *a.e.* on  $\partial\Omega$ , and  $\partial_\nu$  denotes the normal derivative at  $\partial\Omega$ .

In diffuse optical tomography, the inverse problem is to reconstruct the optical absorption distribution  $a_*$  from measurements of the outgoing light intensity on  $\partial\Omega$  given by  $\partial_\nu \Phi_*|_{\partial\Omega}$ , see [9, 27]. It is worth mentioning that, in our diffusion equation model (1.1), if  $l \neq 0$ , then knowing  $\Phi_*$  or  $\partial_\nu \Phi_*$  on  $\partial\Omega$  is mathematically the same.

Diffuse optical tomography produces images with poor accuracy and spatial resolution. It is known to be ill-posed due to the fact that the outgoing light intensities are not very sensitive to local changes of the optical absorption distribution [9, 19, 24, 27]. In [5] we have proposed an original method for reconstructing the optical absorption coefficient by using mechanical perturbations of the medium. While taking optical measurements the medium is perturbed by a propagating acoustic wave. Then cross-correlations between the boundary values of the optical energy density in the medium changed by the propagation of the acoustic wave and those of the optical energy density in the unperturbed one are computed. Finally, under the Born approximation [12], the use of a spherical Radon transform inversion yields a reconstructed image for  $a_*$ , which has a resolution of order the width of the wave front of the acoustic wave propagating in the medium. The Born approximation linearizes the reconstruction problem. It consists of assuming that  $a_*$  is close to a constant and taking the background solution of the diffusion equation for constant optical absorption in place of  $\Phi_*$  [27] as the driving optical energy density at each point in  $\Omega$ .

The idea of mechanically perturbing the medium has been first introduced in [4] for electromagnetic imaging. On the other hand, it is also worth emphasizing that this approach is different from the imaging by controlled perturbations [2, 3, 13, 7, 8, 15, 29], where local changes of the parameters of the medium are produced by focalizing an ultrasound beam. Both techniques lead to resolution enhancements. In imaging by controlled perturbations, the resolution is of order the size of the focal spot while here it is of the order of the width of the wave front of the wave propagating in the medium.

This paper aims to generalize the acousto-optic process behind the Born approximation. We tackle the nonlinear optical reconstruction problem. We develop a mathematical framework for the reconstruction problem in the case where the optical absorption distribution is a perturbation of a piecewise constant function. We introduce an iterative reconstructing algorithm of Landweber-type and prove its convergence and stability. For doing so, we introduce a weak Helmholtz decomposition and interpret in a weak sense the cross-correlation measurements.

To describe our approach, we employ several notations. Each smooth component of  $a_*$  is called an inclusion. The background of  $a_*$  is assumed to be a known positive constant and denoted by  $a_0$ . Assume further the knowledge of a lower bound  $\underline{a}$  and an upper bound  $\bar{a}$  of  $a_*$ , both of which are positive. Finally, let  $D \Subset \Omega$  be known and such that

$$a_* = a_0 \quad \text{in } \Omega \setminus D. \quad (1.2)$$

We next impose some conditions on the unknown inclusions. Let  $k \geq 1$  denote the number of inclusions and  $A_i$  be occupied by the  $i$ th inclusion. Assume:

$I_1$ . for any  $i \in \{1, \dots, k\}$ ,  $A_i$  is a smooth subdomain of  $\Omega$ ,  $\partial A_i$  is connected;

$I_2$ . for any  $j \neq i$ ,  $\overline{A_i} \cap \overline{A_j} = \emptyset$ ;

$I_3$ .  $\cup_{i=1}^k \overline{A_i} \Subset D$ .

All of the assumptions above suggest the definition of the class  $(\mathcal{A}) \subset L^\infty(\Omega)$ , which contains  $a_*$ .

**Definition 1.1** *The function  $a$  is said to belong to class  $(\mathcal{A})$  iff there exist  $k \geq 1$ ,  $A_1, \dots, A_k \Subset D$  satisfying  $I_1, I_2$  and  $I_3$  and  $a_1, \dots, a_k \in \mathcal{C}^2(\overline{A_i}, [\underline{a}, \bar{a}])$  such that*

$$a = \sum_{i=0}^k a_i \mathbf{1}_{A_i}, \quad (1.3)$$

where, again,  $a_0$  was introduced in (1.2),  $A_0 = \Omega \setminus \cup_{i=1}^k \overline{A_i}$ , and  $\mathbf{1}_{A_i}$  denotes the characteristic function of  $A_i$ .

Our main results in this paper can be summarized as follows. A spherical acoustic wave is generated at  $y$  outside  $\Omega$ . Its propagation inside the medium  $\Omega$  changes the optical absorption distribution. Due to the acoustic wave, any point  $x \in \Omega$  moves to its new position  $x + v_{y,r}^\eta(x)$ , where  $v_{y,r}^\eta$  is defined by (3.1) with  $r$  being the radius of the spherical wave impulsion. By linearization, the displacement field is approximately  $v_{y,r}^\eta$  as the thickness  $\eta$  of the acoustic wavefront goes to zero. Hence, the optical absorption of the medium changed by the propagation of the acoustic wave is approximately  $a_*(x + v_{y,r}^\eta)$ , up to an error of order  $\eta$ .

Using cross-correlations between the outgoing light intensities in the medium changed by the propagation of the acoustic wave and those of in the unperturbed one, we get the data  $M_\eta(y, r)$  given by (3.5). In Propositions 4.2 and 4.3, we show that  $M_\eta(y, r)$  converges in the sense of distributions to  $M(y, r)$  as  $\eta \rightarrow 0$ . We refer to  $M(y, r)$  as the ideal data. Making use of a weak Helmholtz decomposition, stated in Lemma 2.6, we relate in Theorem 5.1 the ideal data to the gradient of  $\Phi_*^2 \nabla a_*$ . Since  $a_*$  is piecewise smooth,  $\nabla a_*$  can be defined only in the sense of distributions. Technical arguments and quite delicate estimates are needed in order to establish the fact that the gradient part of  $\Phi_*^2 \nabla a_*$  can be obtained from the cross-correlation measurements using the inverse spherical Radon transform. Based on this, we propose an optimal control approach for reconstructing the values of  $a_*$  inside the inclusions. For doing so, we first detect the support of  $a_* - a_0$  as the support of the gradient part of the data  $\Phi_*^2 \nabla a_*$ . In fact, Lemma 2.6 shows that the support of the data yields the support of the inclusions. Their boundaries are detected as the support

of the discontinuities in the data. Proposition 6.1 provides a Lipschitz stability result for reconstructing piecewise constant optical absorption. In contrast with the recent results in [1, 10, 11], Proposition 6.1 uses only one measurement but the supports of the inclusions are known. Minimizing the discrepancy functional (6.3) we obtain the background constant values of the optical absorption inside the inclusions. Next, in order to recover spatial variations of  $a_*$  inside the inclusions, we minimize the discrepancy between the linear forms  $F[a]$  and  $\Delta\psi$  given by (6.6) and (6.13), respectively. We prove in Theorem 6.3 that the Fréchet derivative of the nonlinear discrepancy functional is well-defined and establish useful estimates as well. We introduce an iterative scheme of Landweber-type for minimizing the discrepancy functional and prove in Theorem 6.5 its convergence provided that the optical absorption coefficient is in the set  $K$  defined by (6.4).

## 2 Preliminaries

### 2.1 Some basic properties

We first recall the following results.

**Proposition 2.1 (weak comparison principle [5])** *Let  $a \in L^\infty(\Omega)$  be a nonnegative function and assume that  $\Phi \in H^1(\Omega)$  satisfies*

$$\begin{cases} -\Delta\Phi + a\Phi & \geq 0 & \text{in } \Omega, \\ l\partial_\nu\Phi + \Phi & \geq 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

*We have  $\Phi \geq 0$  a.e. in  $\Omega$ .*

**Lemma 2.2 (Lemma 4.1 in [5])** *Let  $D$  be as in (1.2) and assume that  $g \in H^{1/2}(\partial\Omega)$  is nonnegative. There exist two positive constants  $\lambda$  and  $\Lambda$  such that for all  $a \in (\mathcal{A})$ , the solution  $\Phi$  of*

$$\begin{cases} -\Delta\Phi + a\Phi = 0 & \text{in } \Omega, \\ l\partial_\nu\Phi + \Phi = g & \text{on } \partial\Omega, \end{cases} \quad (2.2)$$

*satisfies*

$$\lambda \leq \Phi \leq \Lambda \quad \text{in } D. \quad (2.3)$$

**Lemma 2.3 (Lemma 4.2 in [5])** *Let  $T$  be the map that sends  $a \in (\mathcal{A})$  into the unique solution of (1.1) with  $a$  replacing  $a_*$ . Then,  $T$  is Fréchet differentiable. Its derivative at  $a$  is given by*

$$DT[a](h) = \varphi, \quad (2.4)$$

*for  $h \in L^\infty(\Omega)$ , where  $\varphi$  solves*

$$\begin{cases} -\Delta\varphi + a\varphi & = -hT[a] & \text{in } \Omega, \\ l\partial_\nu\varphi + \varphi & = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.5)$$

Moreover,  $DT[a]$  can be continuously extended to  $L^2(\Omega)$  by the same formula given in (2.4) and (2.5) with

$$\|DT[a]\|_{\mathcal{L}(L^2(\Omega), H^1(\Omega))} \leq C\Lambda, \quad (2.6)$$

where  $\Lambda$  is defined in Lemma 2.2 and  $\mathcal{L}(L^2(\Omega), H^1(\Omega))$  is the set of bounded linear operators from  $L^2(\Omega)$  into  $H^1(\Omega)$ .

The following lemma will be helpful to prove the uniqueness of the constructed coefficient. We refer to Appendix A for its proof.

**Lemma 2.4** *Let  $\Omega'$  be the union of several subdomains of  $\Omega$  such that  $\Omega \setminus \Omega'$  is path connected. If  $\phi$  is a bounded solution to*

$$\begin{cases} -\Delta\phi + c\phi &= 0 & \text{in } \Omega \setminus \Omega', \\ l\partial_\nu\phi + \phi &= 0 & \text{on } \partial\Omega, \end{cases} \quad (2.7)$$

for some nonnegative constant  $c$  and  $\partial_\nu\phi \equiv 0$  on  $\partial\Omega$ , then  $\phi \equiv 0$  in  $\Omega \setminus \Omega'$ .

**Corollary 2.5** *Let  $A_0, A_1, \dots, A_k$  be as in Definition 1.1 and let  $a \in (\mathcal{A})$  be defined by such sets. Denote by  $\varphi_j$ ,  $j = 1, \dots, k$ , the solution of*

$$\begin{cases} -\Delta\varphi_j + a\varphi_j &= \mathbf{1}_{A_j}\Phi & \text{in } \Omega, \\ l\partial_\nu\varphi_j + \varphi_j &= 0 & \text{on } \partial\Omega, \end{cases}$$

with  $\Phi$  being the solution of (2.2). Then, the set  $\{\partial_\nu\varphi_j|_{\partial\Omega}\}$  is linearly independent.

*Proof.* Define

$$\varphi = \sum_{j=1}^k \alpha_j \varphi_j \quad \text{in } \Omega,$$

for some  $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ , and assume that  $\partial_\nu\varphi = 0$  on  $\partial\Omega$ . It is obvious that  $\varphi$  is the solution of

$$\begin{cases} -\Delta\varphi + a\varphi &= \sum_{j=1}^k \alpha_j \mathbf{1}_{A_j} \Phi & \text{in } \Omega, \\ l\partial_\nu\varphi + \varphi &= 0 & \text{on } \partial\Omega, \end{cases}$$

and, hence, satisfies (2.7) with  $c = a_0$  and  $\Omega' = \cup_{i=1}^k A_i$ . Thus, by Lemma 2.4,  $\varphi \equiv 0$  in  $\Omega_0$ . On the other hand, for each  $i \in \{1, \dots, k\}$ ,  $\varphi$  solves

$$\begin{cases} -\Delta\varphi + a_i\varphi &= \alpha_i \mathbf{1}_{A_i} \Phi & \text{in } A_i, \\ \varphi &= 0 & \text{on } \partial A_i. \end{cases}$$

We can now apply the strong comparison principle (see, for instance, Lemma 3.1 in [22]) and the Hopf lemma to see that  $\partial_\nu\phi \neq 0$  on  $\partial A_i$ . This contradicts to the fact that  $\phi \equiv 0$  in  $A_0$ .  $\square$

## 2.2 The Helmholtz decomposition in the sense of distributions

The Helmholtz decomposition plays a crucial role in [5] when we established a differential coupling system for  $a$ , where  $a$  was supposed to be in  $\mathcal{C}^2(\overline{\Omega})$ . Fortunately, when  $a$  is no longer smooth but  $\Phi^2 \nabla a$  belongs to  $(H^1(\Omega)^d)^* \subset H^{-1}(\Omega)^d$  for all  $\Phi \in \mathcal{C}^1(\Omega)$ , a corresponding Helmholtz decomposition remains true. Note that for all  $a \in (\mathcal{A})$  and  $\Phi \in \mathcal{C}^1(\Omega)$ ,  $\Phi^2 \nabla a \in (H^1(\Omega)^d)^*$  in the sense that

$$\begin{aligned} \langle \Phi^2 \nabla a, v \rangle_{(H^1(\Omega)^d)^*, H^1(\Omega)^d} &= \langle \Phi^2 \nabla(a - a_0), v \rangle_{(H^1(\Omega)^d)^*, H^1(\Omega)^d} \\ &= - \int_D (a - a_0) \nabla \cdot (\Phi^2 v) dx. \end{aligned} \quad (2.8)$$

The domain of the integral above is written as  $D$  instead of  $\Omega$  because  $a - a_0 = 0$  in  $\Omega \setminus D$ , where  $D$  is introduced in (1.2). By the same reason, we do not require the boundary zero value for the admissible test functions. The last equation in (2.8) suggests that it might be sufficient to impose  $a \in \mathcal{C}^1(\overline{A_i})$ , instead of  $\mathcal{C}^2(\overline{A_i})$ ,  $i = 1, \dots, k$ , as in Definition 1.1. However, we need the differentiability of  $a$  up to second order in each inclusion for some later regularity and estimation purposes.

The following result holds.

**Lemma 2.6** *For any  $U$  in  $H^{-1}(\Omega)^d$  there exist  $\psi \in L^2(\Omega)$  and  $\Psi \in H^{-1}(\Omega)^d$  such that*

$$U = \nabla \psi + \Psi$$

*with  $\nabla \cdot \Psi = 0$ . In particular, if  $U = \Phi^2 \nabla a$  for some  $a \in \mathcal{A}$  then  $\psi$  is continuous and discontinuous at the point where  $a$  is, respectively.*

*Proof.* Letting  $U = (U_1, \dots, U_d) \in H^{-1}(\Omega)^d$ , we denote by  $u = (u_1, \dots, u_d)$  the solution of

$$\begin{cases} -\Delta u &= U & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{cases} \quad (2.9)$$

The vector  $u \in H_0^1(\Omega)^d$  is actually the Riesz representation of  $U$  in  $H_0^1(\Omega)^d$ . Applying the classical Helmholtz decomposition for  $u$  (see, for instance, [14]), we can find  $f \in H^1(\Omega)$  and  $G \in H(\text{curl}, \Omega) := \{w \in L^2(\Omega)^d : \nabla \times w \in L^2(\Omega)^d\}$  such that

$$u = \nabla f + \nabla \times G. \quad (2.10)$$

Here,  $\nabla \cdot G = 0$  inside  $\Omega$  and

$$G \times \nu = 0 \quad \text{on } \partial\Omega. \quad (2.11)$$

Moreover,  $f$  is a solution of

$$\begin{cases} \Delta f &= \nabla \cdot u & \text{in } \Omega, \\ \partial_\nu f &= 0 & \text{on } \partial\Omega. \end{cases} \quad (2.12)$$

Since  $u$  belongs to  $H^1(\Omega)^d$ ,  $\nabla \cdot u \in L^2(\Omega)$ . By standard regularity results, we see that  $f \in H^2(\Omega)$ .

In view of (2.9), taking the Laplacian of (2.10) yields

$$U = \nabla\psi + \Psi,$$

in the sense of distributions, where  $\psi = \Delta f \in L^2(\Omega)$  and  $\Psi$  is divergence free.

We next prove the second statement of the lemma in which  $U = \Phi^2 \nabla a$  for some  $a \in (\mathcal{A})$ . The main tools we use here are the  $H^2$ - and  $\mathcal{C}^1$ -regularity results. Fix  $j \in \{1, \dots, d\}$  and  $i \in \{0, \dots, k\}$ . Denote by  $u_j$  the  $j$ th component of the vector  $u$ , defined in (2.9). Since  $u_j \in H_0^1(\Omega)$ , it belongs to  $H^1(A_i)$ . The function  $u_j$  solves

$$-\Delta u_j = \Phi^2 \partial_{x_j} a, \quad (2.13)$$

in  $A_i$ . Applying Theorem 8.8 in [16], we see that  $u_j$  is in  $H^2(A'_i)$  for all  $A'_i \Subset A_i$ . Hence, differentiating (2.13) gives

$$-\Delta \partial_{x_l} u_j = \partial_{x_l} (\Phi^2 \partial_{x_j} a)$$

in  $A'_i$  for all  $l = 1, \dots, d$ . Since  $\partial_{x_l} u_j \in H^1(A'_i)$  and  $\partial_{x_l} (\Phi^2 \partial_{x_j} a) \in L^2(A'_i)$ , we can apply the  $\mathcal{C}^1$ -regularity result in [20] to see that  $\partial_{x_l} u_j$  is in  $\mathcal{C}^1(A''_i)$  for all  $A''_i \Subset A'_i$ . This implies  $u_j \in \mathcal{C}^2(A_i)$ . Considering the differential equation in (2.12) in each inclusion and following the same regularity process, we see that  $f \in \mathcal{C}^2(A_i)$ . Hence  $\psi = \Delta f$  is continuous in  $A_i$ , which is also the set of continuous points of  $a$ . On the other hand, since  $U = \Phi^2 \nabla a$  involves Dirac distributions supported in  $\cup_i \partial A_i$ ,  $\nabla \cdot u$  is not continuous across  $\cup_i \partial A_i$ , so are  $f$  and  $\psi = \Delta f$ .  $\square$

### 3 The set of data

In this section, we describe the set of data obtained by the acousto-optic process introduced in [4]. The basic idea in order to achieve a resolution enhancement in imaging the optical absorption distribution is as follows. We generate a spherical acoustic wave inside the medium. The propagation of the acoustic wave changes the absorption parameter of the medium. During the propagation of the wave we measure the light intensity on  $\partial\Omega$ . The aim is now to reconstruct the optical absorption coefficient from such set of measurements.

Let  $a \in (\mathcal{A})$  represent the true coefficient  $a_*$ . Let  $S^{d-1}$  be the unit sphere in  $\mathbb{R}^d$ . Let  $\mu > 0$  and let  $S_\mu = \mu S^{d-1}$ , the sphere of radius  $\mu$  and center 0, be such that  $\Omega$  stays inside  $S_\mu$ . We perturb the optical domain  $\Omega$  by spherical acoustic waves generated at point sources  $y \in S_\mu$ . Let  $r \in [r_0, R]$  be the radius of the spherical wave impulsion, where  $r_0$  and  $R$  are the minimum and maximum radii so that the spherical waves generated at point sources on  $S_\mu$  can intersect  $\Omega$ . Let  $\eta \ll 1$  be the acoustic impulsion typical length representing the thickness of the wavefront. Let the position function  $P$  be defined by

$$P : x \mapsto x + v_{y,r}^\eta(x), \quad x \in \Omega,$$

where

$$v_{y,r}^\eta(x) = \eta \frac{r_0}{r} w \left( \frac{r - |x - y|}{\eta} \right) \frac{x - y}{|x - y|}, \quad (3.1)$$

and  $w$  is a smooth function supported on  $[-1, 1]$  with  $\|w\|_\infty = 1$ . Here,  $\|\cdot\|_\infty$  denotes  $\|\cdot\|_{L^\infty([-1, 1])}$ .

In [4], we have shown that the displacement function at the point  $x$  caused by the short diverging spherical acoustic wave generated at  $y$  is given by

$$u_{y,r}^\eta(x) = P^{-1}(x) - x, \quad x \in \Omega. \quad (3.2)$$

Let  $C$  be the cylinder  $S_\mu \times [r_0, R]$ . For each  $(y, r) \in C$ ,  $a_{u_{y,r}^\eta}(x)$  denotes  $a(x + u_{y,r}^\eta(x))$  and  $\Phi_{u_{y,r}^\eta}$  is the optical energy density in the displaced medium, which satisfies

$$\begin{cases} -\Delta \Phi_{u_{y,r}^\eta} + a_{u_{y,r}^\eta} \Phi_{u_{y,r}^\eta} = 0 & \text{in } \Omega, \\ l \partial_\nu \Phi_{u_{y,r}^\eta} + \Phi_{u_{y,r}^\eta} = g & \text{on } \partial\Omega. \end{cases} \quad (3.3)$$

Physically, the outgoing light intensities  $\partial_\nu \Phi|_{\partial\Omega}$  and  $\partial_\nu \Phi_{u_{y,r}^\eta}|_{\partial\Omega}$  are measured. We are thus able to assume the knowledge of the cross-correlation measurements:

$$\frac{1}{\eta^2} \int_{\partial\Omega} g(\partial_\nu \Phi - \partial_\nu \Phi_{u_{y,r}^\eta}) d\sigma, \quad y \in S_\mu, r > 0. \quad (3.4)$$

Integration by parts shows that the quantity above is equal to

$$M_\eta(y, r) = \frac{1}{\eta^2} \int_\Omega (a_{u_{y,r}^\eta} - a) \Phi \Phi_{u_{y,r}^\eta} dx, \quad (3.5)$$

which is considered as our set of data. Here, the coefficient  $1/\eta^2$  is put in front of the integral because both  $\Phi$  and  $\Phi_{u_{y,r}^\eta}$  are bounded (Lemma 2.2) and

$$\|a_{u_{y,r}^\eta} - a\|_{L^1(\Omega)} = O(\eta^2) \quad \text{as } \eta \rightarrow 0^+, \quad (3.6)$$

provided that the following technical condition, named as  $(\mathcal{H})$ , is imposed: there exists  $\delta > 0$  such that for all  $x \in \partial A_i \cap \Sigma_\eta(y, r)$ , either

$H_1$ : the angle formed by the ray  $x - y$  and the normal outward vector of  $A_i$  at  $x$  is greater than  $\delta$ ; or,

$H_2$ : the curvature of  $\partial A_i$  is different to that of the circle or sphere  $\{z \in \mathbb{R}^d : |z - y| = |x - y|\}$  at  $x$  if the angle above is smaller than  $\delta$ .

Here,

$$\Sigma_\eta(y, r) = \{z \in \mathbb{R}^d : r - \eta < |z - y| < r + \eta\}.$$

In fact, this condition guarantees that

$$|A_i \triangle P^{-1}(A_i)| + |A_i \triangle P(A_i)| \leq O(\eta^2). \quad (3.7)$$

Denote

$$V_\varepsilon(S) = \{x \in \mathbb{R}^d, \exists y \in S, |x - y| < \varepsilon\}, \quad (3.8)$$

for any smooth surface  $S$  of  $\mathbb{R}^d$ , and  $\varepsilon > 0$ . Since  $S$  is smooth, the volume of  $V_\varepsilon(S)$  is given by

$$V_\varepsilon(S) = 2\sigma(S)\varepsilon + O(\varepsilon^2).$$



Fix  $(y, \eta) \in C$  and write

$$\begin{aligned} \|a_{u_{y,r}^\eta} - a\|_{L^1(\Omega)} &= \sum_{i=1}^n \int_{A_i \cup P^{-1}(A_i)} |a_{u_{y,r}^\eta} - a| dx \\ &= \sum_{i=1}^n \int_{A_i \cap P^{-1}(A_i)} |a_{u_{y,r}^\eta} - a| dx + \int_{A_i \triangle P^{-1}(A_i)} |a_{u_{y,r}^\eta} - a| dx. \end{aligned} \quad (3.9)$$

As  $u_{y,r}^\eta$  is supported on  $\Sigma_\eta(y, r)$  and  $\|u_{y,r}^\eta\|_\infty = \eta$ ,

$$\begin{aligned} \int_{A_i \cap P^{-1}(A_i)} |a_{u_{y,r}^\eta} - a| dx &= \int_{\Sigma_\eta \cap A_i \cap P^{-1}(A_i)} |a_{u_{y,r}^\eta} - a| dx \\ &\leq \eta \|\nabla a_i\|_{L^\infty(A_i)} |\Sigma_\eta| \\ &\leq \|\nabla a_i\|_{L^\infty(A_i)} \sigma(S(0, R)) \eta^2, \end{aligned}$$

where  $\sigma(S(0, R))$  is the surface measure of the sphere of center  $O$  and radius  $R$ . The second integral in (3.9) is bounded by  $O(\eta^2)$  because of (3.7) and the boundedness of  $a$ .

## 4 The behavior of $M_\eta$ as $\eta$ approaches $0^+$ and the ideal measurements

Consider the open cylinder  $C := S_\mu \times (0, R)$  with its classical product topology.

The construction of

$$\begin{aligned} M_\eta : C &\rightarrow \mathbb{R} \\ (y, r) &\mapsto \frac{1}{\eta^2} \int_{\Omega} (a_{u_{y,r}^\eta} - a) \Phi \Phi_{u_{y,r}^\eta} dx, \end{aligned}$$

has been described in this previous section. The knowledge of this function is obtained from those of  $g$ ,  $\partial_\nu \Phi$  and  $\partial_\nu \Phi_{u_{y,r}^\eta}$  on  $\partial\Omega$ . In this section, we study the limit of  $M_\eta$  as  $\eta \rightarrow 0^+$ . This, together with a weak version of Helmholtz decomposition and the spherical Radon transform, will help us to detect all inclusions.

**Lemma 4.1** *For any  $\eta > 0$ ,  $M_\eta$  is a continuous map on  $C$ .*

*Proof.* It is sufficient to consider only the case  $r > r_0$  because  $M_\eta(y, r) = 0$  for all  $r \leq r_0$  and  $y \in S_\mu$ . Fix  $(y, r) \in S_\mu \times (r_0, R)$  and let  $\{(y_n, r_n)\}_{n \geq 1} \subset S_\mu \times (r_0, R)$  converge to  $(y, r)$ . Noting that  $a_{u_{y,r}^\eta}$  is continuous except on the zero measured set

$$\{x + u_{y,r}^\eta(x) : x \in \cup_{i=1}^n \partial A_i\},$$

we have

$$a(x + u_{y_n, r_n}^\eta(x)) \rightarrow a(x + u_{y, r}^\eta(x))$$

*a.e.* in  $\Omega$ . On the other hand, since  $a$  is uniformly bounded, so is

$$|a(x + u_{y_n, r_n}^\eta(x)) - a(x + u_{y, r}^\eta(x))|^2.$$

It follows by the Lebesgue dominated convergence theorem that

$$a_{u_{y_n, r_n}^\eta} \rightarrow a_{u_{y, r}^\eta} \quad \text{in } L^2(\Omega)$$

as  $n \rightarrow \infty$ . This implies

$$\Phi_{u_{y_n, r_n}^\eta} \rightarrow \Phi_{u_{y, r}^\eta}$$

in both  $H^1(\Omega)$  and  $L^4(\Omega)$ . Note that the  $L^4$  convergence above is valid because  $d$  is either 2 or 3. A direct calculation yields

$$\begin{aligned} & |\eta^2(M_\eta(y_n, r_n) - M_\eta(y, r))| \\ &= \left| \int_{\Omega} [(a_{u_{y_n, r_n}^\eta} - a)\Phi_{u_{y_n, r_n}^\eta} - (a_{u_{y, r}^\eta} - a)\Phi_{u_{y, r}^\eta}] dx \right| \\ &\leq \int_{\Omega} |a_{u_{y_n, r_n}^\eta} - a| |\Phi_{u_{y_n, r_n}^\eta} - \Phi_{u_{y, r}^\eta}| dx + \int_{\Omega} |a_{u_{y_n, r_n}^\eta} - a_{u_{y, r}^\eta}| \Phi_{u_{y, r}^\eta} dx, \\ &\leq 2\bar{a} \|\Phi\|_{L^4(\Omega)} \|\Phi_{u_{y_n, r_n}^\eta} - \Phi_{u_{y, r}^\eta}\|_{L^4(\Omega)} \\ &\quad + \|a_{u_{y_n, r_n}^\eta} - a_{u_{y, r}^\eta}\|_{L^2(\Omega)} \|\Phi\|_{L^4(\Omega)} \|\Phi_{u_{y, r}^\eta}\|_{L^4(\Omega)}. \end{aligned}$$

The lemma follows.  $\square$

Lemma 4.1 guarantees that  $M_\eta$  is measurable. In the case that  $a$  is smooth, which has been studied in [4, 5],  $M_\eta(y, r) \approx \int_{\Omega} \nabla a \cdot u_{y, r}^\eta \Phi^2$  when  $\eta$  is small. However, when  $a$  is piecewise smooth, we need to establish a similar approximation in the weak sense. The following proposition holds. We refer to Appendix B for its proof.

**Proposition 4.2** *Let  $C = S_\mu \times (0, R)$ . For any  $0 < \eta \ll 1$ , define the continuous function*

$$\widetilde{M}_\eta(y, r) = \frac{1}{\eta^2} \int_{\Omega} (a - a_0) \nabla \cdot (\Phi^2 v_{y, r}^\eta) dx, \quad (y, r) \in C, \quad (4.1)$$

where  $\Phi$  is the solution of (1.1) with  $a$  replacing  $a_*$ . Assume  $(\mathcal{H})$  holds and, consequently, (3.7) is valid. Then there exists  $c > 0$ , independent of  $(y, r)$ , such that

$$|M_\eta(y, r) - \widetilde{M}_\eta(y, r)| \leq c\eta, \quad \forall (y, r) \in C. \quad (4.2)$$

It follows from Proposition 4.2 that for each  $(y, r) \in C$ ,

$$\lim_{\eta \rightarrow 0^+} M_\eta(y, r) = \lim_{\eta \rightarrow 0^+} \widetilde{M}_\eta(y, r) := M_{y, r}. \quad (4.3)$$

We cannot expect that  $M$  is a smooth function on  $C$  because  $u_{y, r}^\eta/\eta^2$ , and hence  $v_{y, r}^\eta/\eta^2$ , converges to a distribution supported on the circle (or sphere)  $S(y, r) = \{z : |z - y| = r\}$ . The limit in (4.3) is understood as follows.

Let

$$G(C) = \{f \in L^2(C) : \partial_r f \in L^2(C)\},$$

be a Hilbert space, endowed with the norm

$$\|\cdot\|_{G(C)} = \|\cdot\|_{L^2(C)} + \|\partial_r \cdot\|_{L^2(C)}.$$

Let  $\gamma$  be the (continuous) trace operator from  $C$  to  $S_\mu \times \{0, R\}$  and denote

$$G_0(C) = \gamma^{-1}(0) = \{f \in G(C) : \gamma(f) = 0\}, \quad G^{-1}(C) = G_0(C)^*.$$

We have the following relations

$$H_0^1(C) \subset G_0(C) \subset L^2(C), \quad L^2(C) \subset G^{-1}(C) \subset H^{-1}(C).$$

Let  $\|\cdot\|_1$  denote  $\|\cdot\|_{L^1([-1,1])}$ . The following is the main result of this section. It is a direct consequence of Proposition 4.2.

**Proposition 4.3** *The function  $M_\eta$  converges to the ideal measurements  $M$  in  $G^{-1}(C)$  as  $\eta \rightarrow 0^+$  with*

$$\begin{aligned} \langle M, \varphi \rangle_{(C_0^\infty(C))^*, C_0^\infty(C)} \\ = -r_0 \|w\|_1 \int_{S_\mu} \int_0^R \int_{S^{d-1} \cap \Omega_{y,r}} a(y + r\xi) \frac{\partial}{\partial r} \left( r^{d-2} \Phi^2(y + r\xi) \varphi(y, r) \right) d\xi dr dy \end{aligned} \quad (4.4)$$

where

$$\Omega_{y,r} = \left\{ \frac{x-y}{r} : x \in \Omega \right\}.$$

*Proof.* For any  $\varphi$  in  $G_0(C)$ , we have

$$\begin{aligned} \langle \widetilde{M}_\eta, \varphi \rangle &= - \int_{y \in S_\mu} \int_{r=0}^R \int_{\Omega} (a - a_0)(x) \nabla \cdot_x \left( \Phi^2(x) \frac{v_{y,r}^\eta(x)}{\eta^2} \varphi(y, r) \right) dx dr dy \\ &= - \int_{y \in S_\mu} \int_{\Omega} (a - a_0)(x) \nabla \cdot_x \left( \Phi^2(x) \int_{r=0}^R \frac{v_{y,r}^\eta(x)}{\eta^2} \varphi(y, r) dr \right) dx dy. \end{aligned}$$

Then by the change of variables  $x = y + \rho\xi$  we get

$$\frac{v_{y,r}^\eta(x)}{\eta^2} = \frac{r_0}{r\eta} w\left(\frac{\rho-r}{\eta}\right) \xi.$$

Hence we can write

$$\begin{aligned} \langle \widetilde{M}_\eta, \varphi \rangle &= - \\ &\int_{S_\mu} \int_{S^{d-1}} \int_{\rho=0}^R (a - a_0)(y + \rho\xi) \frac{\partial}{\partial \rho} \left( \rho^{d-1} \Phi^2(y + \rho\xi) \int_{r=0}^R \frac{r_0}{r\eta} w\left(\frac{\rho-r}{\eta}\right) \varphi(y, r) dr \right) d\rho d\xi dy \end{aligned}$$

Since

$$\frac{1}{\eta} w\left(\frac{\rho-r}{\eta}\right) \xrightarrow{\eta \rightarrow 0} \|w\|_1 \delta_\rho,$$

we deduce that

$$\int_{r=0}^R \frac{r_0}{r\eta} w\left(\frac{\rho-r}{\eta}\right) \varphi(y, r) dr \xrightarrow{\eta \rightarrow 0} \frac{\|w\|_1 r_0}{\rho} \varphi(y, \rho),$$

and then

$$\begin{aligned} \langle \widetilde{M}_\eta, \varphi \rangle &\xrightarrow{\eta \rightarrow 0} \\ &- \|w\|_1 r_0 \int_{S_\mu} \int_{\rho=0}^R \int_{S^{d-1}} a(y + \rho\xi) \frac{\partial}{\partial \rho} \left( \rho^{d-2} \Phi^2(y + \rho\xi) \varphi(y, \rho) \right) d\rho d\xi dy, \end{aligned}$$

as desired.  $\square$

## 5 Detecting the inclusions

Using the fact that  $\Phi^2 \nabla a \in (H^1(\Omega)^d)^* \subset H^{-1}(\Omega)^d$ , we can employ Lemma 2.6 to write that

$$\Phi^2 \nabla a = \nabla \psi + \Psi, \quad (5.1)$$

where  $\Psi$  is a divergence free field and  $\psi \in L^2(\Omega)$ . Since that both  $\Phi^2 \nabla a$  and  $\nabla \psi$  are in  $(H^1(\Omega)^d)^*$ , so is  $\Psi$ . Moreover, it follows from the usual integration by parts formula and the boundary condition (2.11) that

$$\langle \Psi, \nabla v \rangle = 0, \quad \forall v \in \mathcal{C}^\infty(\overline{\Omega}). \quad (5.2)$$

For a distribution  $f \in (\mathcal{C}_0^\infty(C))^*$ , we define its spherical Radon transform  $\mathcal{R}[f]$  in the sense of distributions by

$$\langle \mathcal{R}[f], \varphi \rangle_{(\mathcal{C}_0^\infty(C))^*, \mathcal{C}_0^\infty(C)} = \langle f, \mathcal{R}^*[\varphi] \rangle_{(\mathcal{C}_0^\infty(C))^*, \mathcal{C}_0^\infty(C)},$$

where

$$\mathcal{R}^*[\varphi](x) = \int_C \varphi(y, |x - y|) dy \quad \text{for } \varphi \in \mathcal{C}_0^\infty(C).$$

We have the following result.

**Theorem 5.1** *The spherical Radon transform  $\mathcal{R}[\psi]$  of  $\psi$  satisfies the equation*

$$M = r_0 \|w\|_1 r^{d-2} \frac{\partial \mathcal{R}[\psi]}{\partial r} \quad (5.3)$$

*in the sense of distributions.*

*Proof.* Let  $\varphi \in \mathcal{C}_0^\infty(C)$ , and for a fixed  $y \in S_\mu$  we define

$$F_y(x) = \varphi(y, |x - y|) \frac{x - y}{|x - y|^2} \quad x \in \Omega.$$

For any  $y \in S_\mu$ , the vector  $F_y$  is in  $H^1(\Omega)^d$  because  $|x - y| \geq r_0$ . Equation (5.1) yields

$$\langle \Phi^2 \nabla a, F_y \rangle = \langle \nabla \psi, F_y \rangle + \langle \Psi, F_y \rangle$$

and since  $F_y$  is the gradient of the function given by

$$x \longmapsto \int_0^{|x-y|} \frac{\varphi(y, \rho)}{\rho} d\rho,$$

it follows from (5.2) that  $\langle \Psi, F_y \rangle = 0$ . Here,  $\langle \cdot, \cdot \rangle$  denotes the duality pair between  $H^1(\Omega)^d$  and  $(H^1(\Omega)^d)^*$ . Therefore,

$$\langle \Phi^2 \nabla a, F_y \rangle = \langle \nabla \psi, F_y \rangle. \quad (5.4)$$

A simple calculation shows

$$\begin{aligned} \langle \Phi^2 \nabla a, F_y \rangle &= \int_\Omega a \nabla \cdot (\Phi^2 F_y) dx \\ &= \int_0^R \int_{S^{d-1} \cap \Omega_{y,r}} [a \nabla \cdot (\Phi^2 F_y)] (y + r\xi) r^{d-1} d\xi dr, \end{aligned}$$

and hence,

$$\langle \Phi^2 \nabla a, F_y \rangle = \int_0^R \int_{S^{d-1} \cap \Omega_{y,r}} a(y + r\xi) \frac{\partial}{\partial r} \left[ \Phi^2(y + r\xi) \varphi(y, r) r^{d-2} \right] d\xi dr. \quad (5.5)$$

Combining (4.4), (5.4), and (5.5) implies

$$\begin{aligned} \langle M, \varphi \rangle &= \|w\|_{L^1(\Omega)} \int_{S_\mu} \langle \nabla \psi, F_y \rangle dy \\ &= -\|w\|_{L^1(\Omega)} \int_{S_\mu} \int_{\Omega} \psi \nabla \cdot (F_y) dx dy \\ &= -\|w\|_{L^1(\Omega)} \int_{S_\mu} \int_0^R \int_{S^{d-1} \cap \Omega_{y,r}} [\psi \nabla \cdot (F_y)] (y + r\xi) r^{d-1} d\xi dr dy \\ &= -\|w\|_{L^1(\Omega)} \int_{S_\mu} \int_0^R \int_{S^{d-1} \cap \Omega_{y,r}} \psi(y + r\xi) \frac{\partial}{\partial r} \left[ \varphi(y, r) r^{d-2} \right] d\xi dr dy \\ &= -r_0 \|w\|_{L^1(\Omega)} \int_{S_\mu} \int_0^R \mathcal{R}[\psi](y, r) \frac{\partial}{\partial r} \left[ \varphi(y, r) r^{d-2} \right] dr dy \\ &= r_0 \|w\|_{L^1(\Omega)} \langle r^{d-2} \frac{\partial \mathcal{R}[\psi]}{\partial r}, \varphi \rangle, \end{aligned}$$

and the proof is complete.  $\square$

**Remark 5.2** *Theorem 5.1 provides the knowledge of the derivative of the spherical Radon transform of  $\psi$  (see Appendix C for the reconstruction of  $\mathcal{R}[\psi]$  from its derivative). Note that the function  $\psi$  itself can be reconstructed in a stable way from  $\mathcal{R}[\psi]$  using an inversion (filtered) retroprojection formula for the spherical Radon transform. From this, all inclusions are detected by the second statement in Lemma 2.6, noticing that  $\partial A_i$  is the set of discontinuous points of  $\psi$ .*

## 6 A reconstruction algorithm of the true coefficient

With all inclusions  $A_1, A_2, \dots, A_k$  in hand, we are able to find an initial guess for  $a_*$  using the unique continuation property (Lemma 2.4) and then employ a Landweber type iteration to reconstruct  $a_*$ . As an initial guess, we reconstruct constant values inside each inclusion by minimizing the discrepancy between computed and measured boundary data. We prove a Lipschitz stability result for the reconstruction of the optical absorption coefficient in the class of piecewise constant distributions provided that the support of the inclusions is known.

### 6.1 The data of boundary measurements and an initial guess

Define

$$\mathcal{S} = \left\{ \sum_{i=0}^k \alpha_i \mathbf{1}_{A_i} : \alpha_0 = a_0 \text{ and } \alpha_1, \dots, \alpha_k \in [\underline{a}, \bar{a}] \right\}.$$

Let  $a_1$  and  $a_2$  be in  $\mathcal{S}$ . Their difference can be written as

$$a_2 - a_1 = \sum_{i=1}^k h_i \mathbf{1}_{A_i},$$

for some  $h = (h_1, \dots, h_k) \in B = [\underline{a} - \bar{a}, \bar{a} - \underline{a}]^k$ . Note that  $B$  can be considered as a closed ball of  $\mathbb{R}^k$  with respect to the  $\infty$ -norm of  $\mathbb{R}^k$  given by

$$|h| = \max\{|h_1|, \dots, |h_k|\}.$$

The compactness of  $B$  plays an important role in our analysis. Suppose that  $l \neq 0$ . Denote by  $\Phi_1$  and  $\Phi_2$  the optical energy density functions that correspond to  $a_1$  and  $a_2$ . The function  $\phi = \Phi_1 - \Phi_2$  solves

$$\begin{cases} -\Delta\phi + a_1\phi &= \sum_{i=1}^k h_i \mathbf{1}_{A_i} \Phi_2 & \text{in } \Omega, \\ l\partial_\nu\phi + \phi &= 0 & \text{on } \partial\Omega. \end{cases} \quad (6.1)$$

Using  $\phi$  as the test function in the variational form of (6.1), we see that

$$\int_{\Omega} (|\nabla\phi|^2 + \underline{a}\phi^2) dx + l \int_{\partial\Omega} (\partial_\nu\phi)^2 d\sigma \leq |h|\lambda \int_{\Omega} |\phi| dx,$$

where  $\lambda$  is defined in Lemma 2.2. This implies

$$\|\partial_\nu\phi\|_{L^2(\partial\Omega)} \leq C|h|$$

and, therefore, the continuity of the map  $h \mapsto \partial_\nu\phi|_{\partial\Omega}$ . Since the map  $h \in \partial_{\mathbb{R}^k}B \mapsto \|\partial_\nu\phi\|_{L^2(\partial\Omega)}$  is continuous and nonzero (due to Corollary 2.5), we can employ the compactness of  $\partial_{\mathbb{R}^k}B$  in  $\mathbb{R}^k$  to see that

$$c(a_1) = \min_{h \in \partial_{\mathbb{R}^k}B} \|\partial_\nu\phi\|_{L^2(\partial\Omega)} > 0.$$

Identifying  $\mathcal{S}$  with a compact subset of  $\mathbb{R}^k$ , we can conclude that

$$c = \inf_{a_1 \in \mathcal{S}} c(a_1) > 0.$$

Properly scaling the inequality

$$\|\partial_\nu\phi\|_{L^2(\partial\Omega)} \geq c$$

for all  $h \in \partial_{\mathbb{R}^k}B$ , we arrive at the following Lipschitz stability result using only one measurement. Note here that the support of the inclusions is known and only the value of the optical absorption coefficient inside each inclusion is to determine.

**Proposition 6.1** *There exists  $c > 0$  such that for all  $a_1, a_2 \in \mathcal{Q}$ ,*

$$\|\partial_\nu\Phi_1 - \partial_\nu\Phi_2\|_{L^2(\partial\Omega)} \geq c\|a_1 - a_2\|_{L^\infty(\Omega)}, \quad (6.2)$$

where  $\Phi_1$  and  $\Phi_2$  are the solutions of (1.1) with  $a_*$  replaced by  $a_1$  and  $a_2$ , respectively.

**Remark 6.2** *Inequality (6.2) guarantees the uniqueness of the reconstruction for  $a_* \in \mathcal{S}$  if  $\partial_\nu \Phi_*|_{\partial\Omega}$  is considered as the data given. It, moreover, implies the stability in the sense that small noise does not cause large error.*

Proposition 6.1 suggests us to minimize the quadratic misfit functional:

$$J(a) = \frac{1}{2} \|\partial_\nu \Phi - \partial_\nu \Phi_*\|_{L^2(\partial\Omega)}^2, \quad (6.3)$$

where  $a$  varies in  $\mathcal{S}$  and  $\Phi_*$  is the true optical energy density. This is possible since  $\mathcal{S}$  is identical with a compact subset of  $\mathbb{R}^k$ . By (6.2), the function  $a_I = \operatorname{argmin} J$  is close to  $a_*$  provided that  $a_*$  is a perturbation of a constant on each inclusion  $A_i$ . Therefore,  $a_I$  can be considered as the background constant optical absorption distribution in the inclusions. For simplicity, we propose the following exhaustion method: for each fine partition  $P$  of the interval  $[\underline{a}, \bar{a}]$ , try all values of  $\alpha_i$  such that  $\alpha_i$  equals each element of  $P$ , and finally choose the  $k$ -tuple  $(\alpha_1, \dots, \alpha_k)$  that gives the smallest  $\|\partial_\nu \Phi - \partial_\nu \Phi_*\|_{L^2(\partial\Omega)}$ .

## 6.2 Internal data map and its differentiability

Define the set that  $a_* = (a_1^*, \dots, a_k^*)$ , identifying with true optical absorption coefficient  $a_*$  of the form (1.3), belongs to

$$K := \{a \in \prod_{j=1}^k W_0^{1,4}(A_j) : \underline{a} \leq a_i \leq \bar{a} \text{ and } \|\nabla a_i\|_{L^4(A_j)} \leq \theta, i = 1, \dots, k\}, \quad (6.4)$$

where  $\theta$  will be determined later in (6.8). It is obvious that  $K$  is closed and convex in  $H$  where  $H = \prod_{j=1}^k H_0^1(A_j)$  is a Hilbert space with the usual inner product

$$\langle u, v \rangle_H = \sum_{i=1}^k \int_{A_j} \nabla u_j \cdot \nabla v_j dx$$

for all  $u = (u_1, \dots, u_k)$  and  $v = (v_1, \dots, v_k)$  in  $H$ .

Now, let the map  $F : K \rightarrow H^*$  be defined as follows. For all  $(a_1, \dots, a_k) \in K$ , let

$$a = \sum_{i=0}^k a_i \mathbf{1}_{A_i}, \quad (6.5)$$

and

$$F[a](v) = \sum_{j=1}^k \int_{A_j} T[a]^2 \nabla a_j \cdot \nabla v \quad \text{for all } v \in H, \quad (6.6)$$

where  $T[a]$  was defined in Lemma 2.3. We call  $F$  the *internal data map*.

**Theorem 6.3** *The map  $F$  is Fréchet differentiable in  $K$  and*

$$DF[a](h, v) = \sum_{i=1}^k \int_{A_i} (2T[a]DT[a](h)\nabla a_i + T[a]^2 \nabla h_i) \nabla v_j dx \quad (6.7)$$

for all  $a = (a_1, \dots, a_k) \in K$ ,  $h = (h_1, \dots, h_k) \in \prod_{j=1}^k W_0^{1,4}(A_j) \cap L^\infty(A_j)$  and  $v = (v_1, \dots, v_k) \in H$ . Assume further

$$0 < \theta < \frac{C_{\Omega'} \lambda^2}{\Lambda^2}, \quad (6.8)$$

where  $\Omega' = \cup_{j=1}^k A_j$  and  $C_{\Omega'}$  is the norm of the embedding map of  $H^1(\Omega')$  into  $L^4(\Omega')$ , multiplied with the constant in (2.6). Then,  $DF[a]$  is well-defined on  $H$  and there exists a positive constant  $C$  such that for all  $h \in H$ ,

$$\|DF[a](h)\|_{H^*} \geq C\|h\|_H. \quad (6.9)$$

**Remark 6.4** The term  $DT[a](h)$  in (6.7) is understood as  $DT[a]$  acting on the function that is equal to 0 in  $A_0$  and to  $h_j$  in  $A_j$ ,  $j = 1, \dots, k$ .

*Proof of Theorem 6.3.* The Fréchet differentiability of  $F$  and the expression (6.7) of  $DF$  can be deduced from Lemma 2.3 and the standard rules in differentiation. We only prove (6.9). In fact, for all  $h \in H$ ,

$$\begin{aligned} DF[a](h, h) &= \sum_{j=1}^k \int_{A_j} (T[a]^2 |\nabla h_j|^2 + 2T[a]DT[a](h) \nabla a_j \nabla h_j) dx \\ &\geq \sum_{j=1}^k \left[ \int_{A_j} (T[a]^2 |\nabla h_j|^2) dx - \int_{A_j} |2T[a]DT[a](h) \nabla a_j \nabla h_j| dx \right] \\ &\geq \lambda^2 \left( \|h\|_H^2 - \sum_{j=1}^k \frac{\Lambda}{\lambda^2} \|DT[a](h)\|_{L^4(A_j)} \|\nabla a_j\|_{L^4(A_j)} \|\nabla h_j\|_{L^2(A_j)} \right). \end{aligned}$$

It follows from the continuous embedding of  $H^1(D)$  into  $L^4(D)$  and (2.6) that

$$DF[a](h, h) \geq \lambda^2 \left( 1 - \frac{C_D \Lambda^2 \theta}{\lambda^2} \right) \|h\|_{H_0^1(D)}^2,$$

and therefore, inequality (6.9) holds true.  $\square$

We now make use of Theorem 6.3 in order to prove a local Landweber condition which guarantees the convergence of the reconstruction algorithm.

Let  $a$  and  $a'$  be in  $K$ . We can find  $t \in [0, 1]$  such that

$$\|F[a] - F[a']\|_{H^*} = \|DF[ta + (1-t)a'](a - a')\|_{H^*} \geq C\|a - a'\|_H \quad (6.10)$$

by (6.9). Hence, if  $\|a - a'\|_H$  is small enough, then

$$\|F[a] - F[a'] - DF[a](a - a')\|_{H^*} \leq \eta \|F[a] - F[a']\|_{H^*} \quad (6.11)$$

for some  $\eta < \frac{1}{2}$ . In other words,  $F$  satisfies the local Landweber condition (see [17]).



### 6.3 Landweber iteration

Going back to equation (5.1), we have

$$\nabla \cdot \Phi^2 \nabla a = \Delta \psi \quad (6.12)$$

in the sense of distributions. However, the equation above can be understood in the classical sense in each inclusion  $A_i$ . This observation plays an important role in reconstructing the true coefficient from the initial guess given in Subsection 6.1.

Considering  $\Delta \psi$  as an element of  $H^*$  defined by

$$-\Delta \psi(v) = \sum_{j=1}^k \int_{A_j} \nabla \psi \cdot \nabla v_j dx, \quad (6.13)$$

for all  $v = (v_1, \dots, v_k)$ , we rewrite (6.12) as

$$F[a] = \Delta \psi. \quad (6.14)$$

Recalling that  $K$  is closed and convex in  $H$ , we can employ the classical Hilbert projection theorem to define the projection from  $H$  onto  $K$  as

$$P : H \ni h \mapsto \operatorname{argmin}\{\|h - a\|_H : a \in K\}. \quad (6.15)$$

It is not hard to verify that

$$\|P(h) - a\|_H \leq \|h - a\|_H \quad (6.16)$$

for all  $a \in K$ .

We next solve (6.14) using the Landweber method to minimize

$$I(a) = \frac{1}{2} \|F[a] - \Delta \psi\|_{H^*}^2,$$

where  $a$  varies in  $K$  with the initial guess  $a_I = (\alpha_1, \dots, \alpha_k)$ , obtained in Subsection 6.1. The corresponding guess for the coefficient is

$$a_I = \sum_{i=1}^k \alpha_i \mathbf{1}_{A_i}.$$

There is a gap if we minimize  $I$  by the classical Landweber sequence given by

$$\begin{aligned} a^{(0)} &= a_I, \\ a^{(n+1)} &= a^{(n)} - \mu DF[a^{(n)}]^*(F[a^{(n)}] - \Delta \psi) \end{aligned}$$

because  $a^{(1)}$  may not belong to  $K$  and  $F[a^{(1)}]$  is not well-defined. Motivated by (6.16), which implies  $P(a^{(n)})$  is closer to  $a_*$  than  $a^{(n)}$  is, we modify this formula as

$$a^{(n+1)} = P(a^{(n)}) - \mu DF[P(a^{(n)})]^*(F[P(a^{(n)})] - \Delta \psi). \quad (6.17)$$

We have the following convergence result.

**Theorem 6.5** *Suppose that the true optical distribution  $a_* \in K$ . Let  $a^{(n)}$  be defined by (6.17) with  $a^{(0)}$  being the initial (piecewise constant) guess obtained as the minimizer of (6.3). Then the sequence  $a^{(n)}$  converges in  $H$  to  $a_*$  as  $n \rightarrow \infty$ .*

Noting that  $F$  satisfies the local Landweber condition (see (6.11)), we can repeat the proof of Proposition 2.2 in [17] to see that

$$\|a^{(n+1)} - a_*\|_H^2 + (1 - 2\eta)\|F[P(a^{(n)}) - \Delta\psi]\|_{H^*}^2 \leq \|P(a^{(n)}) - a_*\|_H^2.$$

This and (6.16) imply

$$\|P(a^{(n+1)}) - a_*\|_H^2 - \|P(a^{(n)}) - a_*\|_H^2 \leq (2\eta - 1)\|F[P(a^{(n)}) - \Delta\psi]\|_{H^*}^2 \leq 0. \quad (6.18)$$

It follows that

$$\sum_{i=1}^{\infty} \|F[P(a^{(i)})] - \Delta\psi\|_{H^*}^2 \leq \frac{1}{1 - 2\eta} \|a_*\|_H^2,$$

and hence

$$F[P(a^{(n)})] \rightarrow \Delta\psi \text{ in } H_0^1(\Omega) \text{ as } n \rightarrow \infty. \quad (6.19)$$

On the other hand, we can see from (6.18) that the sequence  $(P(a^{(n)}))_{n \geq 1}$  is bounded in  $H$ . Assume that  $P(a^{(n)})$  converges weakly to  $a'$  for some  $a' \in H$ . Since  $K$  is closed and convex, it is weakly closed and therefore  $a' \in K$ . Passing to a subsequence if necessary, this sequence converges to  $a'$  *a.e.* and also converges strongly to  $a'$  in  $\prod_{j=1}^k L^2(A_j)$ . So,  $T[P(a^{(n)})]$  converges to  $T[a']$  in  $H^1(\Omega)$  and hence in  $L^4(\Omega)$ . For all  $v \in H$ , we have

$$\begin{aligned} & \sum_{j=1}^k \int_{A_j} (T[P(a^{(n)})]^2 \nabla P(a^{(n)}) - T[a']^2 \nabla a') \nabla v dx \\ &= \sum_{j=1}^k \left[ \int_{A_j} (T[P(a^{(n)})]^2 - T[a']^2) \nabla P(a^{(n)}) \nabla v dx + \int_{A_j} T[a']^2 (\nabla P(a^{(n)}) - \nabla a') \nabla v dx \right], \end{aligned}$$

which goes to 0 by the dominated convergence theorem and the weak convergence of  $P(a^{(n)})$  to  $a'$  in  $H$ . We have obtained  $F[a'] = \Delta\psi = F[a_*]$ . Using (6.9) gives  $a' = a_*$ .

In summary, if the true coefficient  $a_*$  is a perturbation of a constant on each inclusion then the coefficient  $a_I$  obtained in Section 6.1 is quite closed to  $a_*$ . Moreover, the misfit between the initial guess  $a_I$  and the true distribution  $a_*$  can be properly corrected by the sequence in (6.17).

## 7 Concluding remarks

In this paper we have introduced a Landweber scheme for reconstructing piecewise smooth optical absorption distributions from opto-acoustic measurements and proved its convergence. Because of the jumps in the absorption coefficient, we have used weak formulations for the Helmholtz decomposition for  $\Phi_*^2 \nabla a_*$  and the relation between the spherical Radon transform of its gradient part  $\psi$  and the cross-correlation measurements  $M_\eta(y, r)$ . Note

that we can enrich the set of data as follows. For  $f \in L^2(\partial\Omega)$  such that  $f \geq 0$  a.e. on  $\partial\Omega$ , compute instead of (3.4) the quantity

$$M_\eta^{f,g}(y, r) = \frac{1}{\eta^2} \int_{\partial\Omega} (f \partial_\nu \Phi_{u_{y,r}^\eta}^g - g \partial_\nu \Phi^f) d\sigma, \quad y \in S_\mu, r > 0.$$

Similarly to (3.5), integration by parts yields

$$M_\eta^{f,g}(y, r) = \frac{1}{\eta^2} \int_\Omega (a_{u_{y,r}^\eta} - a) \Phi^f \Phi_{u_{y,r}^\eta}^g dx, \quad (7.1)$$

where  $\Phi^f$  is the solution of (1.1) with  $g$  replaced by  $f$ .

The enriched data (7.1) may be used in order to generalize our approach to the case of measurements of the outgoing light intensities on only part of  $\partial\Omega$  by choosing  $f$  supported only on the accessible part of the boundary. Another interesting and challenging problem is to prove statistical stability of the proposed reconstruction with respect to a measurement noise by combining Fourier techniques together with statistical tools [8]. Numerical implementation of the Landweber-type iteration is under consideration and will be the subject of a forthcoming publication. The behavior of the proposed method with respect to the optical absorption contrast will be investigated. It is expected that higher the contrast, more efficient the method is.

## A Proof of Lemma 2.4

The boundedness of  $\phi$  together with the assumption that  $\phi \equiv 0$  on  $\partial\Omega$  imply by standard regularity results that  $\phi \in \mathcal{C}^1(\partial\Omega \cup \Omega \setminus \overline{\Omega'})$ . Arguing similarly to Proposition 2.5 in [5], we see that  $\phi \in \mathcal{C}^2(\Omega \setminus \overline{\Omega'})$ . Define

$$\mathcal{U} = \{x \in \Omega \setminus \overline{\Omega'} : u(x) \neq 0\}.$$

The continuity of  $\phi$  shows that  $\mathcal{U}$  is open. Assume, on contrary, that  $\mathcal{U}$  is nonempty.

Noting that  $\mathcal{U}$  can be decomposed as the union of its connected open subsets. Denote by  $\mathcal{O}$  the connected component of  $\mathcal{U}$ , which is closest to  $\partial\Omega$ . Without loss of generality, assume that  $\phi > 0$  in  $\mathcal{O}$ . Let

$$\delta = \text{dist}(\mathcal{O}, \partial\Omega).$$

The distance above is understood as the length of the shortest curve, contained in  $\Omega \setminus \overline{\Omega'}$  and connecting  $\overline{\mathcal{O}}$  and  $\partial\Omega$ .

In the case that  $\delta = 0$ ,  $\partial\mathcal{O}$  and  $\partial\Omega$  have a common point  $x_0$ . Applying the Hopf lemma for the equation

$$\begin{cases} -\Delta\phi + c\phi &= 0 & \text{in } \mathcal{O}, \\ \phi &> 0 & \text{on } \partial\mathcal{O}, \end{cases}$$

gives  $\partial_\nu \phi(x_0) < 0$ , which is impossible.

When  $\delta > 0$ , it is easy to see that  $\phi \equiv 0$  in a neighbourhood of  $\partial\Omega$ . Assume that such a neighbourhood and  $\mathcal{O}$  have a common boundary point  $x_0$ . Noting that  $\nabla\phi(x_0) = 0$ , we can apply the Hopf lemma again to get the contradiction.  $\square$

## B Proof of Proposition 4.2

We write  $u$  and  $v$  when referring to  $u_{y,r}^\eta$  and  $v_{y,r}^\eta$  respectively for simplicity. Using (3.7) and the same arguments when estimating  $\|a_u - a\|_{L^1(\Omega)}$  in the previous section yields

$$\|a_u - a\|_{L^2(\Omega)} \leq O(\eta).$$

This, together with standard  $H^2$ -regularity results (see, for instance, [16, Theorems 8.8 and 8.12]) and the embedding of  $H^2(\Omega)$  into  $L^\infty(\Omega)$ , gives

$$\|\Phi_u - \Phi\|_{L^\infty(\Omega)} \leq O(\eta).$$

Hence, it follows from (3.6) that

$$\begin{aligned} \left| \int_{\Omega} (a_u - a) \Phi \Phi_u dx - \int_{\Omega} (a_u - a) \Phi^2 dx \right| &\leq \int_{\Omega} \Phi |a_u - a| |\Phi_u - \Phi| dx \\ &\leq \|\Phi\|_{L^\infty(\Omega)} \|a_u - a\|_{L^1(\Omega)} \|\Phi_u - \Phi\|_{L^\infty(\Omega)} \\ &\leq c\eta^3. \end{aligned} \tag{B.1}$$

The constant  $c$  depends only on  $\bar{a} = \max a$  and  $\underline{a} = \min a$ , both of which are assumed to be known. The independence of  $c$  on  $\|\Phi\|_{L^\infty(\Omega)}$  can be deduced from Lemma 2.2. Now, note that the second integral in the left hand side of (B.1) can be rewritten as

$$\begin{aligned} \int_{\Omega} (a_u - a) \Phi^2 dx &= \sum_{i=1}^n \int_{A_i \cup P(A_i)} (a_u - a) \Phi^2 dx \\ &= \sum_{i=1}^n \left[ \int_{A_i \cap P(A_i)} (a_u - a) \Phi^2 dx + \int_{A_i \triangle P(A_i)} (a_u - a) \Phi^2 dx \right], \end{aligned}$$

and that the integral in (4.1) is equal to

$$\begin{aligned} \int_{\Omega} (a - a_0) \nabla \cdot (\Phi^2 v) dx &= \sum_{i=1}^n \int_{A_i} (a - a_0) \nabla \cdot (\Phi^2 v) dx \\ &= \sum_{i=1}^n \left[ \int_{\partial A_i} (a_i - a_0) \Phi^2 v \cdot \nu_i d\sigma - \int_{A_i} \Phi^2 \nabla a \cdot v dx \right] \\ &= \sum_{i=1}^n \left[ \int_{\partial A_i} (a_i - a_0) \Phi^2 v \cdot \nu_i d\sigma \right. \\ &\quad \left. - \int_{A_i \cap P(A_i)} \Phi^2 \nabla a \cdot v dx - \int_{A_i \setminus P(A_i)} \Phi^2 \nabla a \cdot v dx \right]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\left| \int_{\Omega} (a_u - a) \Phi^2 dx + \int_{\Omega} (a - a_0) \nabla \cdot (\Phi^2 v) dx \right| \\ &\leq \sum_{i=1}^n \left| \int_{A_i \cap P(A_i)} (a_u - a + \nabla a \cdot v) \Phi^2 dx \right| + \left| \int_{A_i \setminus P(A_i)} \Phi^2 \nabla a \cdot v dx \right| \\ &\quad + \left| \int_{A_i \triangle P(A_i)} (a_u - a) \Phi^2 dx - \int_{\partial A_i} (a_i - a_0) \Phi^2 v \cdot \nu_i d\sigma \right|. \end{aligned}$$

Denote by  $\alpha_i$ ,  $\beta_i$  and  $\gamma_i$  the last three quantities in the inequality above. We need to prove that they all are bounded by  $O(\eta^3)$  to complete the proof.

- (i) Since for all  $i = 1, \dots, k$ ,  $a_i \in \mathcal{C}^2(\overline{A_i})$  and  $\|D^2 a_i\|_{L^\infty(A_i)}$  are bounded by some known constants, we can find a constant  $c$  such that

$$\left| \int_{A_i \cap P(A_i)} (a_u - a - \nabla a \cdot u) \Phi^2 dx \right| \leq \|\Phi\|_{L^\infty(A_i)}^2 \|D^2 a\|_{L^\infty(A_i)} \eta^2 |\Sigma_\eta| \leq c_1^i \eta^3.$$

On the other hand, using the classical substitution method in integration gives

$$\begin{aligned} & \left| \int_{A_i \cap P(A_i) \cap \Sigma_\eta} \nabla a \cdot (u + v) \Phi^2 dx \right| \\ & \leq \left| \int_{S(y,r)} (\Phi^2 \nabla a)(\xi) \cdot \int_{-\eta}^{\eta} (\mathbf{1}_{A_i}(u + v)) \left( \left(1 + \frac{\rho}{r}\right) \xi \right) d\rho d\xi \right| \\ & \quad + \left| \int_{S(y,r)} \int_{-\eta}^{\eta} \left[ (\Phi^2 \nabla a) \left( \left(1 + \frac{\rho}{r}\right) \xi \right) - (\Phi^2 \nabla a)(\xi) \right] \cdot (\mathbf{1}_{A_i}(u + v)) \left( \left(1 + \frac{\rho}{r}\right) \xi \right) d\rho d\xi \right| \\ & \leq 0 + 2\eta^2 \|\partial_r(\Phi^2 \nabla a)\|_{L^\infty(A_i)} |\Sigma_\eta| \\ & \leq c\eta^3. \end{aligned}$$

The quantity  $\alpha_i$  is bounded from above by  $O(\eta^3)$  because

$$\alpha_i \leq \left| \int_{A_i \cap P(A_i)} (a_u - a - \nabla a \cdot u) \Phi^2 dx \right| + \left| \int_{A_i \cap P(A_i) \cap \Sigma_\eta} \nabla a \cdot (u + v) \Phi^2 dx \right|.$$

- (ii) The fact that  $\beta_i \leq O(\eta^3)$  can be deduced from the boundedness of the integrand and (3.7).
- (iii) The main point of the proof is the estimate of  $\gamma_i$ . Denote  $\tilde{v}(z) = v(z)/|v(z)|$  when it is defined and

$$\begin{aligned} \chi : \partial\mathcal{A}_i \times [0, \eta] &\longrightarrow A_i \triangle P(A_i) \\ (z, t) &\longmapsto z + t\tilde{v}(z). \end{aligned}$$

$\chi$  is well defined when  $v(z)$  is non-zero and not parallel to  $\partial A_i$ . For any  $z \in \partial\mathcal{A}_i$  satisfying this condition, we denote  $T(z)$  the tangent plane to  $\partial A_i$  and  $B(z)$  a basis adapted to the sum  $\mathbb{R}^d = T(z) \oplus \mathbb{R}\tilde{v}(z)$ . Let  $I_{d-1}$  denote the  $(d-1) \times (d-1)$  identity matrix. We get

$$d\chi(z, t) = \begin{bmatrix} I_{d-1} + t d\tilde{v}(z) & 0 \\ * & 1 \end{bmatrix}$$

and as  $\tilde{v}(z) = (z - y)/|z - y|$ . The operator  $d\tilde{v}(z)$  does not depend on  $\eta$  and  $t d\tilde{v}(z) = O(\eta)$  with a constant depending on  $r_0$ . Then,

$$\det(d\chi(z, t)) = 1 + t \nabla \cdot (\tilde{v})(z) + O(\eta^2) = 1 + O(\eta).$$

As  $B(z)$  is not orthonormal, the differential volume written with the variables  $(z, t)$  depends on the angle between  $\tilde{v}(z)$  and  $\nu(z)$  called  $\theta(z)$ . This volume at the point  $z + t\tilde{v}(z)$  is  $(1 + O(\eta)) \cos(\theta(z)) dt dz$ . Knowing this, we denote

$$(\partial A_i)^\pm = \{z \in \partial A_i, \pm \theta(z) > 0\}$$

and write

$$\begin{aligned} \int_{P(A_i) \setminus A_i} (a_u - a) \Phi^2 dx &= \int_{(\partial A_i)^+} \int_0^{|v(z)|} (a_u - a_0) \Phi^2(z + t\tilde{v}(z)) (1 + O(\eta)) \\ &\quad \times \cos(\theta(z)) dt dz \end{aligned}$$

and as  $a_i$  and  $\Phi$  are  $\mathcal{C}^1(\overline{A_i})$ , we can write that for any  $z \in (\partial A_i)^+$ , and  $t \in [0, |v(z)|]$ ,

$$|(a_u - a_0) \Phi^2(z + t\tilde{v}(z)) - (a_i - a_0) \Phi^2(z)| \leq O(\eta).$$

Then,

$$\left| \int_0^{|v(z)|} (a_u - a_0) \Phi^2(z + t\tilde{v}(z)) (1 + O(\eta)) dt - (a_i - a_0) \Phi^2(z) |v(z)| \right| \leq O(\eta^2).$$

Now, noticing that  $\cos(\theta)|v(z)| = v(z) \cdot \nu(z)$  and that  $\sigma((\partial A_i)^+ \cap \Sigma_\eta)$ , the surface measure of  $(\partial A_i)^+ \cap \Sigma_\eta$ , is of order  $O(\eta)$ , we have

$$\left| \int_{P(A_i) \setminus A_i} (a_u - a) \Phi^2 - \int_{(\partial A_i)^+} (a_i - a_0) \Phi^2 v \cdot \nu \right| \leq O(\eta^3).$$

We also get

$$\left| \int_{A_i \setminus P(A_i)} (a_u - a) \Phi^2 - \int_{(\partial A_i)^-} (a_i - a_0) \Phi^2 v \cdot \nu \right| \leq O(\eta^3)$$

by the same arguments. □

## C Construction of $\mathcal{R}[\psi]$ from formula (5.3)

In order to construct  $\mathcal{R}[\psi]$  from formula (5.3), we need to invert the operator  $\frac{\partial}{\partial r} : L^2(C) \longrightarrow G^{-1}(C)$  and prove the stability of the inversion. For any  $f \in L^2(C)$ , by Fubini's theorem, the function  $F(y, r) = \int_0^r f(y, \rho) d\rho$  is well-defined and in  $G(C)$  but not in  $G_0(C)$ . Since this operator is acting on distributions which are zero on  $S_\mu \times ]0, r_0[$ , we introduce

$$\begin{aligned} p : L^2(C) &\longrightarrow G_0(C) \\ \varphi &\longmapsto \left[ (y, r) \mapsto - \int_0^r \left( \varphi(y, \rho) - \frac{R}{r_0} \chi_{]0, r_0[}(\rho) \varphi(y, \rho R/r_0) \right) d\rho \right] \end{aligned}$$

and its dual

$$p^* : G^{-1}(C) \longrightarrow L^2(C). \tag{C.1}$$

The following result holds.

**Proposition C.1** *For all  $f \in L^2(C)$  such that  $f = 0$  on  $S_\mu \times ]0, r_0[$ , we have the inversion formula*

$$p^*\left[\frac{\partial f}{\partial r}\right] = f.$$

*Proof.* For any  $\varphi \in L^2(C)$ , we have  $\frac{\partial}{\partial r}p[\varphi] = -\varphi$  on  $S_\mu \times [r_0, R[$  and therefore,

$$\int_C p^*\left[\frac{\partial f}{\partial r}\right] \varphi = \left\langle \frac{\partial f}{\partial r}, p[\varphi] \right\rangle_{G^{-1}(C), G_0^1(C)} = - \int_C f \frac{\partial}{\partial r}p[\varphi] = \int_C f \varphi,$$

which yields the claimed result.  $\square$

**Proposition C.2** *For all  $u \in \mathcal{M} := \{v \in G^{-1}(C) : \text{supp}(v) \subset S_\mu \times [r_0, R[ \}$ ,*

$$\|p^*u\|_{L^2(C)} \leq \|u\|_{G^{-1}(C)}$$

*Proof.* We first note that  $p^*[u] = 0$  on  $S_\mu \times ]0, r_0[$ . Then, for any  $\varphi \in L^2(C)$ , we get

$$\begin{aligned} \left| \int_C p^*[u] \varphi \right| &= \left| \int_C p^*[u] \chi_{[r_0, R[} \varphi \right| \leq \|u\|_{G^{-1}(C)} \|p[\chi_{[r_0, R[} \varphi]\|_{G_0^1(C)} \\ &\leq \|u\|_{G^{-1}(C)} \left\| \frac{\partial}{\partial r} p[\chi_{[r_0, R[} \varphi] \right\|_{L^2(C)} \\ &\leq \|u\|_{G^{-1}(C)} \|\chi_{[r_0, R[} \varphi\|_{L^2(C)} \\ &\leq \|u\|_{G^{-1}(C)} \|\varphi\|_{L^2(C)}, \end{aligned}$$

and the proof is complete.  $\square$

Finally, we deduce the following result.

**Corollary C.3** *From formula (5.3), we have*

$$\mathcal{R}[\psi] = \frac{1}{r_0 \|w\|_1} p^*(r^{d-2} M).$$

Moreover, for  $\eta$  small, if  $\mathcal{R}[\psi_\eta] = \frac{1}{r_0 \|w\|_1} p^*(r^{d-2} M_\eta)$ , then

$$\|\mathcal{R}[\psi - \psi_\eta]\|_{L^2(C)} \leq \frac{R^{d-2}}{r_0 \|w\|_1} \|M - M_\eta\|_{G^{-1}(C)},$$

which insures the stability of the construction of  $\mathcal{R}[\psi]$  from the measurements  $M_\eta$ .

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